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Application of Kemmer-β Formalism to Spin-$\frac{1}{2}$ Systems

M. SAMIULLAH*

International Centre for Theoretical Physics, Trieste, Italy

and

K. G. KHALILI

Department of Physics, University of Azarabadgan, Tabriz, Iran

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Abstract

The nonphysical components of Kemmer-β formalism for spin-$\frac{1}{2}$ particles have been removed. It is shown that in the presence of an external electromagnetic field the formalism is identical with that of Dirac's second-order equation and thus is solvable in specific cases by the usual methods

1. *Introduction*

It is well known that Kemmer-β fields for spin 0 and 1 have redundant or nonphysical components. As such, Sakata & Taketani (1940) and, later on, Heitler (1943) reformulated the theory in terms of physical components by isolating the nonphysical ones. A few years ago Aragone (1967) discussed the old case of spin-1 massive vector mesons together with the spin-2 fields in connection with the removal on nonphysical components. In the present paper we have removed the nonphysical components of Kemmer-β formalism for spin-$\frac{1}{2}$ particles (Barut & Samiullah, 1960). It is interesting to remark that the equation for spin-$\frac{1}{2}$ particles reduces to the second-order Dirac equation in the presence of an external electromagnetic field.

The extension of Kemmer-β formalism to spin-$\frac{1}{2}$ particles was done by Barut & Samiullah (1960) several years ago. The motivation of such an extension was to propose a formalism having the feature that, in attempts of unifying and classifying the elementary particle interactions, one describes all free particles by the same formalism, the difference appearing only in the interaction terms.

* On leave of absence from Department of Physics, University of Azarabadgan, Tabriz, Iran.

2. *Kemmer-β Algebra*

In order to fix up our notation and to indicate the particular representation which we have used throughout this paper, in the following we briefly sketch the Kemmer algebra.

The β matrices satisfy the following commutation relations:

$$\beta^\mu \beta^\nu \beta^\sigma + \beta^\sigma \beta^\nu \beta^\mu = \beta^\mu g^{\nu\sigma} + \beta^\sigma g^{\nu\mu} \tag{1}$$

and no further specification of the β^μ is needed. There are 126 linearly independent quantities in Kemmer algebra generated by β^μ. The algebra is semisimple and has three irreducible representations of dimensionalities 1, 5, and 10.

In our work we have used the following 5 x 5 irreducible representations:

$$\beta^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \beta^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\beta^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \beta^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{2}$$

It is customary to define a quantity η in β formalism as follows:

$$\eta = 2\beta^{02} - 1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & +1 \end{pmatrix} \tag{3}$$

3. *Kemmer-β Formalism for Spin-$\frac{1}{2}$ Particles*

The equation for spin-$\frac{1}{2}$ particles given by Barut & Samiullah (1960) is as follows:

$$[\beta^\mu P_\mu + m - (e/2m)\Lambda]\,\psi = 0 \tag{4}$$

where

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^{\mu\nu} F_{\mu\nu} \end{pmatrix}$$

The zeros being 2 x 2 zero matrices:

The effect of an external field in Eq. (4) is twofold: in $P_\mu \Rightarrow P_\mu - eA_\mu$ and in the additional term $(-e/2m)\,\Lambda$ which is characteristic for spin-$\frac{1}{2}$ particles. For further details we refer the reader to the original paper by Barut et al. (1960).

4. *Removal of Nonphysical Components*

To remove the nonphysical components from the theory we proceed as follows.

We write the Lagrangian density for this field as

$$\mathscr{L} = (i/z)(\bar{\psi}\beta^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\beta^\mu\psi) - \bar{\psi}m\psi + (e/2m)\bar{\psi}\Lambda\psi \tag{5}$$

with $\bar{\psi} = \psi^\dagger\eta$.

We use $c = \hbar = 1, g^{00} = -g^{11} = -g^{22} = -g^{33} = 1, g^{\mu\nu} = 0$ for $\mu \neq \nu$. The Euler–Lagrange equation with the independent variations of ψ and $\bar{\psi}$ yields, respectively, the following equations:

$$[-i\beta^\mu\partial_\mu + m - (e/2m)\,\Lambda]\,\psi = 0 \tag{6}$$

$$\bar{\psi}[i\beta^\mu\overline{\partial}_\mu + m - (e/2m)\,\Lambda] = 0 \tag{7}$$

We define the canonical momenta as

$$\pi = \frac{\partial\mathscr{L}}{\partial\dot{\psi}} = \frac{i}{2}\,\bar{\psi}\beta^0, \qquad \bar{\pi} = \frac{\partial\mathscr{L}}{\partial\dot{\bar{\psi}}} = -\frac{i}{2}\beta^0\psi \tag{8}$$

We further define the projection operators

$$P_1 = \tfrac{1}{2}(1+\eta), \qquad P_2 = \tfrac{1}{2}(1-\eta) \tag{9}$$

and express a 10-dimensional identity operator as

$$I = P_1 + P_2 \tag{10}$$

The field ψ can be expressed as

$$\psi = P_1\psi + P_2\psi = \psi_1 + \psi_2, \qquad \psi_1 = P_1\psi, \qquad \psi_2 = P_2\psi \tag{11}$$

We notice that ψ_1 has four components equal in number to twice the required ones for a spin-$\frac{1}{2}$ field and ψ_2 has six components.

As, in the absence of any external field, Eq. (4) reduces to a 5-component equation, the "bare" particle part, i.e., Eq. (4) without the Λ-containing term, has the correct number of components. However, in the presence of external fields the number of components is doubled. Thus ψ_1 would be the physical part and ψ_2 its nonphysical one.

In conformity with the idea of building the theory in terms of only the physical part, in view of Eq. (8) we redefine the canonical momenta as

$$\pi = (i/2)\,\bar{\psi}_1\beta^0, \qquad \bar{\pi} = -(i/2)\,\beta^0\psi_1 \tag{12}$$

By the nature of the β representation Eq. (2), it is clear that β^0 induces a one-to-one linear transformation in the 4-dimensional space obtained by the action of the projection operator P_1 on ψ.

Multiplying Eq. (6) by P_2 on the left and Eq. (7) on the right and making use of the relations

$$(1-\eta)\beta^0 = 0 \qquad (1 \mp \eta)\beta^k = \beta^k(1 \pm \eta) \tag{13}$$

we find

$$\psi_2 = -(i/m)\beta^k \partial_k \psi_1, \qquad \bar{\psi}_2 = (i/m)\partial_k \bar{\psi}_1 \beta^k \tag{14}$$

Thus the Hamiltonian density in terms of the momenta defined in Eq. (12) takes the form

$$\mathscr{H}' = \pi \partial_0 \psi_1 + \partial_0 \bar{\psi}_1 \bar{\pi} - \mathscr{L} \tag{15}$$

Inserting for π and $\bar{\pi}$ from Eq. (12) and for $\mathscr{L}$ from Eq. (5), substituting from Eq. (11) $\psi = \psi_1 + \psi_2$ and $\bar{\psi} = \bar{\psi}_1 + \bar{\psi}_2$, recalling

$$\beta^0 P_1 = 0 = P_2 \beta^0, \qquad P_1 P_2 = 0 = P_2 P_1 \tag{16}$$

using Eq. (13), eliminating the spatial derivatives of ψ_2 and $\bar{\psi}_2$ which appear in $\mathscr{H}'$ and using the fact that one can ignore the divergences in a density, we obtain the following relation:

$$\mathscr{H}' = \bar{\psi}_1 [m - (e/2m)\Lambda] \psi_1 - i(\bar{\psi}_2 \beta^k \partial_k \psi_1 - \partial_k \bar{\psi}_1 \beta^k \psi_2) \tag{17}$$

For neglecting divergences, we express as an example a term like $\partial_k \bar{\psi}_2 \beta^k \psi_2$ as follows:

$$\partial_k \bar{\psi} \beta^k \psi_2 = \partial_k(\bar{\psi}_2 \cdot \beta^k \psi_2) - \bar{\psi}_2 \beta^k \partial_k \psi_2 \tag{18}$$

and neglect the divergence term.

Further, using Eq. (14) and $\bar{\psi}_1 = \psi_1{}^\dagger \eta$, the Hamiltonian density reduces to the following form:

$$\mathscr{H}' = \psi_1{}^\dagger m \psi_1 + (1/m)[\beta^k \partial_k \psi_1]^\dagger [\beta^k \partial_k \psi_1] - (e/2m)\psi_1{}^\dagger \Lambda \psi_1 \tag{19}$$

Furthermore, using the definitions of canonical momenta from Eq. (12), $\mathscr{H}'$ can be written as a function of ψ, $\bar{\psi}_1$, π, and $\bar{\pi}$.

We find that the right-hand side of Eq. (19) multiplied by β^{02} can be expressed as

$$\begin{aligned}\beta^{02}[\bar{\psi}_1 m \psi_1 &+ (1/m)[\beta^k \partial_k \psi_1]^\dagger [\beta^k \partial_k \psi_1] - (e/2m)\bar{\psi}_1 \Lambda \psi_1] \\ &= -im[\pi \beta^0 \psi_1 - \bar{\psi}_1 \beta^0 \bar{\pi}] - (i/m)[\partial_l \pi]\beta^0 \beta^l \beta^k \partial_k \psi_1 \\ &\quad - \partial_l \bar{\psi}_1 \beta^l \beta^k \beta^0 \partial_k \bar{\pi} + (ie/2m)[\pi \beta^0 \Lambda \psi_1 - \bar{\psi}_1 \Lambda \beta^0 \bar{\pi}].\end{aligned} \tag{20}$$

Equation (20) forces us to redefine our Hamiltonian density as $\beta^{02}\mathscr{H}'$, and to regard β^{02} as a unity operator in this theory. In what follows we designate $\beta^{02}\mathscr{H}'$ as $\mathscr{H}$ and write our new Hamiltonian density as

$$\mathscr{H} = -im(\pi\beta^0\psi_1 - \bar{\psi}_1\beta^0\bar{\pi}) - (i/m)[(\partial_l\pi)\beta^0\beta^l\beta^k\partial_k\psi_1 - \partial_l\bar{\psi}_1\beta^l\beta^k\beta^0\partial_k\bar{\pi}] + (ie/2m)(\pi\beta^0\Lambda\psi_1 - \bar{\psi}_1\Lambda\beta^0\bar{\pi}) \qquad (21)$$

Now with the redefinition of the Hamiltonian density Eq. (21), we can recast Eq. (19) as

$$\mathscr{H} = \bar{\psi}_1\beta^0 m\psi_1 + (1/m)\bar{\psi}_1 P_k P_l \beta^0\beta^k\beta^l\psi_1 - \bar{\psi}_1(e/2m)\beta^0\Lambda\psi_1 \qquad (22)$$

or the Hamiltonian $H \equiv \int \mathscr{H} dV$, which for the normalized field function will be the expectation value of the operators

$$\beta^0 m + (1/m)P_k P_l \beta^0\beta^k\beta^l - (e/2m)\beta^0\Lambda$$

that is

$$H = \langle \beta^0 m + (1/m)P_k P_l \beta^0\beta^k\beta^l - (e/2m)\beta^0\Lambda \rangle \qquad (23)$$

In a proper basis where the operator is diagonalized, the eigenvalue equation for H will be completely solvable, and in case the operator is Hermitian it yields real eigenvalues which are physically measurable quantities.

In view of the representation of the β matrices, Eq. (2), in the presence of an external field, H reduces to

$$H = \langle eA_0\beta^{02} + \beta^0 m + (p^2/m)\beta^0\beta^{32} - (e/2m)\beta^0\beta^{32}\sigma^{\mu\nu}F_{\mu\nu} \rangle \qquad (24)$$

In a basis where H is diagonalizable, its eigenvalues, i.e., the energy spectrum E of a spin-$\frac{1}{2}$ particle under the influence of an external field, can be expressed as the eigenvalue of the right-hand side of Eq. (24). Recall that $(e/2)F_{\mu\nu}\sigma^{\mu\nu} = \boldsymbol{\sigma}\cdot\mathbf{H} - i\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}$, where

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\boldsymbol{\sigma} = \rho_1\boldsymbol{\sigma}$$

and $\boldsymbol{\epsilon}$ and $\mathbf{H}$ are the electric and the magnetic field strengths, respectively. An explicit construction of H in the matrix form reads as follows:

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & eA_0 & \dfrac{P^2}{m} + m + \dfrac{1}{m}(\boldsymbol{\sigma}\cdot\mathbf{H} - i\rho_1\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}) \\ 0 & 0 & 0 & m & eA_0 \end{bmatrix} \qquad (25)$$

Thus the equation yielding eigenvalues E of H is given as follows:

$$[P^2 - e^2A_0{}^2 + (\boldsymbol{\sigma}\cdot\mathbf{H} - i\rho_1\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}) + (m^2 - E^2) - 2eA_0E]\psi = 0 \qquad (26)$$

We remark that this is the second-order Dirac equation in the presence of an external electromagnetic field and can be solved in specific cases by the usual methods.

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